Selected topics on the functional renormalization group and its applications

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Outline

• Functional RG approach to QFTs
  • Perturbative
  • Wilsonian (non perturbative)

• Multicritical Yukawa theories

• Applications to Hamiltonian systems:
  • Quantum mechanics
  • RFT for Regge limit of QCD

• Conclusions
Introduction

Physical systems, very different at microscopic level, can show phases characterized by the same Universal behavior when the correlation length diverges (2nd order phase transition).

Critical phenomena are conveniently described by Quantum and Statistical Field Theories.

Most famous example:

3D Ising universality class (Magnetic systems, Water) in a Landau-Ginzburg description as a scalar QFT,

\[ S = -J \sum_{\langle ij \rangle} s_i s_j + B \sum_i s_i \]

\[ s_i = \pm 1 \]

RG is the proper tool to investigate related questions
Critical theories

Theory space
(fields and symmetries)

The critical theories are points in a suitable theory space characterized by scale invariance. If there is Poincare’ invariance it is often lifted to conformal invariance.
RG

In a Renormalization Group description critical field theories are associated to fixed points of the flow, where scale invariance is realized.

- These fixed points may control the IR behavior of the theories.
  (example: Wilson-Fisher fixed point)

- Fundamental physics in a QFT description require renormalizability conditions which in the most general case goes under the name of Asymptotic Safety: existence of a fixed point with a finite number of UV attractive directions. Asymptotic freedom is a particular case with a gaussian fixed point.

Common formulations:

- Perturbation theory in presence of small parameters, e.g. $\varepsilon$-expansion below the critical dimension

- Wilsonian non perturbative, exact equations but not solvable in practice. (Polchinski and Wetterich/Morris equations)
The main constraints are given by the field content and the symmetries, but this leaves still too many possible theories for a generic dimension $d$.

It is therefore useful to start from some kind of Landau-Ginzburg description to single out some possible solutions.

- This is the starting point for an RG analysis.

**Couplings are coordinates in theory space, spanned by a basis of operators**

- The points corresponding to critical theories may be CFT fixed by the **Conformal data**: the scaling dimensions of the primary operators and the structure constants defining their 3 point correlators. No lagrangian formulation is required.
Universal data and RG

How to get in an RG framework informations on the critical theory?
If conformal, the so called conformal data?

- For example in the perturbative $\varepsilon$-expansion approximation using the universal beta function coefficients, e.g. in a massless $\overline{\text{MS}}$ scheme

Critical quantities are encoded in the expansion coefficients describing the flow around the scale invariant point: $\beta^i(g_\ast) = 0$

$$\beta^k(g_\ast + \delta g) = \sum_i M^k_i \delta g^i + \sum_{i,j} N^k_{ij} \delta g^i \delta g^j + O(\delta g^3)$$

$$M^i_j \equiv \left. \frac{\partial \beta^i}{\partial g^j} \right|_\ast, \quad N^i_{jk} \equiv \frac{1}{2} \left. \frac{\partial^2 \beta^i}{\partial g^j \partial g^k} \right|_\ast$$

Moving to a diagonal basis in the linear sector

$$\sum_{i,j} S^a_i M^i_j (S^{-1})^j_b = -\theta_a \delta^a_b$$
Universal data and RG

\[ \theta_a = d - \Delta_a \]

\[ \tilde{C}_{bc} = \sum_{i,j,k} S^a_{i} N^i_{jk} (S^{-1})^j_b (S^{-1})^k_c \]

RG flow seen along the eigendirections around the fixed point up to second order

\[ S = S_* + \sum_a \mu^a \lambda^a \int d^d x \mathcal{O}_a(x) + O(\lambda^2). \]

\[ \beta^a = -(d - \Delta_a) \lambda^a + \sum_{b,c} \tilde{C}_{bc}^a \lambda^b \lambda^c + O(\lambda^3). \]

one can extract not only the scaling dimensions, but also, reversing an argument from Cardy for a CFT, some OPE coefficients (structure constants) at order \( O(\varepsilon) \)

\[ \langle \mathcal{O}_a(x) \mathcal{O}_b(y) \cdots \rangle = \sum_c \frac{1}{|x - y|^{\Delta_a + \Delta_b - \Delta_c}} C^c_{ab} \langle \mathcal{O}_c(x) \cdots \rangle \]

Linear term coefficients transform homogeneously
Quadratic term coefficients transform inhomogeneously

Possible scheme dependence!
Leading counterterms in perturbation theory at order $g^2$, dim reg $\overline{\text{MS}}$

$$
\mathcal{L}_{\text{c.t.}} = \frac{1}{\epsilon} \frac{1}{2(4\pi)^2} (12g)^2 \phi^4 - \frac{1}{\epsilon} \frac{1}{6(4\pi)^4} (4!)^2 (\partial \phi)^2
$$

Rescaling the coupling: $g \rightarrow (4\pi)^2 g$

Beta function: $\beta_g = -\epsilon g + 72g^2$

Two fixed points:

- $g_* = 0$
- $g_* = \frac{\epsilon}{72}$

UV gaussian IR Wilson-Fisher

Anomalous dimension: $\eta = 2\tilde{\gamma}_1 = 96g^2$

$\eta$ is a universal quantity, independent from any coupling reparameterization!
**Functional perturbative RG example: Ising UC**

How to study **deformations** around the Wilson-Fisher fixed point? \( d = 4 - \epsilon \)

**Couplings:**
\[
\mathcal{L} = \frac{1}{2} (\partial \phi)^2 + g_1 \phi + g_2 \phi^2 + g_3 \phi^3 + g_4 \phi^4
\]

\[
\beta_1 = 12 g_2 g_3 - 108 g_3^2 - 288 g_2 g_3 g_4 + 48 g_1 g_4^2
\]

**Dimensionful beta functions**
(global rescaling as before)
\[
\begin{align*}
\beta_2 &= 24 g_4 g_2 + 18 g_3^2 - 1080 g_3^2 g_4 - 480 g_2 g_4^2 \\
\beta_3 &= 72 g_4 g_3 - 3312 g_3 g_4^2 \\
\beta_4 &= 72 g_4^2 - 3264 g_4^3
\end{align*}
\]

**At functional level:**
\[
\mathcal{L} = \frac{1}{2} Z(\phi)(\partial \phi)^2 + V(\phi)
\]

**1 loop** \( \beta_V = \frac{1}{2} \eta \phi V^{(1)} + a \frac{(V^{(2)})^2}{(4\pi)^2} + b \frac{V^{(2)}(V^{(3)})^2}{(4\pi)^4} + \cdots \) \( \quad a = \frac{1}{2} \)

\[
\begin{align*}
\beta_Z &= \eta Z + \frac{1}{2} \eta \phi Z^{(1)} + c \frac{(V^{(4)})^2}{(4\pi)^4} + \cdots \\
\end{align*}
\]

**2 loop** \( \quad b = -\frac{1}{2} \)

**Take home message**

\[ c = -\frac{1}{6} \]
FPRG for multicritical models

Landau-Ginzburg lagrangian

\[ S[\phi] = \int d^d x \left\{ \frac{1}{2} (\partial \phi)^2 + \mu \left( \frac{m}{2} - 1 \right) \epsilon \frac{g}{m!} \phi^m \right\} \]

Upper critical dimension

\[ d = d_m - \epsilon \]

One marginal interaction at \( d_m \).

Study of deformations: we limit to a truncation

\[ \mathcal{L} = \frac{1}{2} Z(\phi)(\partial \phi)^2 + V(\phi) \]


RG, odd Codello, Safari, G.P.V., Zanusso EPJ C78 (2018) 1

Multi-loop diagrams at functional level: \( m = 2n \)

LO: \hspace{1cm} NLO: 

\[ \beta_v = - d \phi + \frac{d - 2 + \eta}{2} \phi \phi' + \frac{n - 1}{n!} \frac{c^{n-1}}{4} \phi^{(n)}(\phi)^2 - \frac{n - 1}{16 n!} c^{2n-2} \Gamma(\delta_n) \sum_{r+s+t=n} \frac{K^r_{rs t}}{r! s! t!} \phi^{(r+s)}(\phi) \phi^{(s+t)}(\phi) \phi^{(t+r)}(\phi) \]

\[ - \frac{(n-1)^2}{16 n!} c^{2n-2} \sum_{s+t=n} \frac{n - 1 + L^r_{n}}{s! t!} \phi^{(n)}(\phi) \phi^{(n+s)}(\phi) \phi^{(n+t)}(\phi) \]

\[ \beta_z = \eta z(\phi) + \frac{d - 2 + \eta}{2} \phi \phi' - \frac{(n-1)^2 c^{2n-2}}{(2n)!} \frac{1}{4} \phi^{(2n)}(\phi)^2 + \frac{n - 1}{n!} \frac{c^{n-1}}{2} \left[ z^{(n)}(\phi) \phi^{(n)}(\phi) + z^{(n-1)}(\phi) \phi^{(n+1)}(\phi) \right] \]

Rescaling functions and fields to dimensionless quantities \( v(\phi), z(\phi) \)
FPRG for multicritical models

General pattern of mixing of the operators present in the truncation

\[ V : \begin{pmatrix} 1 & \phi & \phi^2 & \cdots & \phi^{2n-1} & \phi^{2n} & \cdots & \phi^{4n-3} & \phi^{4n-2} & \cdots \end{pmatrix} \]

\[ Z : \begin{pmatrix} \phi \partial \phi & \phi^2 \partial \phi & \cdots & \phi^{2n-3} \partial \phi & \phi^{2n-2} \partial \phi & \cdots \end{pmatrix} \]

\[ W_1 : \begin{pmatrix} \phi^2 \partial^2 \phi \end{pmatrix} \]

\[ W_2 : \begin{pmatrix} (\partial \mu \partial \nu \partial \sigma \partial \phi)^2 \end{pmatrix} \]

\[ W_3 : \begin{pmatrix} (\Box \phi)^2 \end{pmatrix} \]

\[
\begin{pmatrix}
M^{(0)} & M^{(2)} \\
M^{(2)} & M^{(4)} \\
& \cdots
\end{pmatrix}
\]

Anomalous dimensions of composite operators

\[
\tilde{\gamma}_i = \frac{2(n-1)n!}{(2n)!} \frac{i!}{(i-n)!} \epsilon
\]

\[
\tilde{\omega}_i = \frac{2(n-1)n!}{(2n)!} \frac{(i+1)!}{(i-n+1)!} \epsilon
\]

At leading order the stability matrix \(\mathbf{M}\) is triangular.

\[
\hat{\gamma}_i = \frac{i \eta}{2} + \frac{(n-1)i!}{(i-n)!} \frac{2n!}{(2n)!} \left[ \epsilon - \frac{n}{n-1} \eta \right] + 2n \eta \delta_i^{2n}
\]

\[
+ \frac{(n-1)i!n!^6}{(2n)!^2} \Gamma(\delta_n) \sum_{r+s+t=2n, r,s,t \neq n} \frac{K_{rst}^n}{(r!s!t!)^2} \left[ \frac{2n!}{3(i-n)!} - \frac{r!}{(i-2n+r)!} \right] \epsilon^2
\]

\[
+ \frac{(n-1)^2i!n!^5}{(2n)!^2} \sum_{s+t=n} \frac{n-1 + L_{st}^n}{(s!t!)^2} \left[ \frac{1}{(i-n)!} - \frac{2s!}{n!(i-2n+s)!} \right] \epsilon^2.
\]

OPE coefficients are read off the quadratic expansion of the beta functions.
Other studies

• Multicritical higher derivative theories: there can be many marginal operators at criticality, results still to be understood in CFT.

• Shift symmetric theories
  Safari, G.P.V. in preparation

Multifield theories

• Potts models (cubic)
  Osborn, Stergiou arXiv:1707.06165

• Potts models (quintic)
  Codello, Safari, G.P.V., Zanusso in preparation
  Non trivial in d=3

Perturbative $\epsilon$-expansion useful guide towards non perturbative regimes.
Non perturbative functional RG flows

Perturbation theory is very powerful to derive some qualitative informations even for infinite set of universal data, but for strongly interacting theories non perturbative tools are needed.

- Wilsonian flows:

  require the partition function to be independent from a UV cutoff. In general one can have

\[
\Lambda \frac{d}{d\Lambda} e^{-S_{\Lambda}[\varphi]} = \int dx \frac{\delta}{\delta \varphi(x)} \left( \psi^\Lambda_x [\varphi] e^{-S_{\Lambda}[\varphi]} \right)
\]

\[
\Lambda \frac{d}{d\Lambda} S_{\Lambda}[\varphi] = \int dx \left( \frac{\delta S_{\Lambda}[\varphi]}{\delta \varphi(x)} \psi^\Lambda_x [\varphi] - \frac{\delta \psi^\Lambda_x [\varphi]}{\delta \varphi(x)} \right)
\]

In general the flow induced by coarse-graining corresponds to a non trivial action-dependent field redefinition

\[
\varphi'(x) = \varphi(x) - \frac{\delta \Lambda}{\Lambda} \psi^\Lambda_x [\varphi]
\]
Wilson–Polchinski RG flows

\[ Z_{\Lambda_0}[J] = \int [d\varphi] \, e^{-\frac{1}{2} \varphi \cdot \Delta^{-1} \cdot \varphi - S^I_{\Lambda_0}[\varphi] + J \cdot \varphi} \]

Split in low (L) and high (H) energy modes

\[ \varphi = \varphi_L + \varphi_H \]
\[ \Delta = \Delta_L + \Delta_H \]
\[ \varphi_L \text{ has support roughly for } |p| < \Lambda \]

Integrating the high energy modes one defines the interacting action \( S^I_\Lambda \) from

\[ e^{W_\Lambda[\varphi_L, J]} = Z_\Lambda[\varphi_L, J] = e^{-\frac{1}{2} J \cdot \Delta_H \cdot J + J \cdot \varphi_L - S^I_\Lambda[\Delta_H \cdot J + \varphi_L]} \]

It is flowing according to the Polchinski equation

\[ \Lambda \frac{d}{d\Lambda} S^I_\Lambda[\varphi] = \frac{1}{2} \int dx dy \left( -\Lambda \frac{d}{d\Lambda} \Delta_H \right)^{xy} \left[ \frac{\delta S^I[\varphi]}{\delta \varphi(y)} \frac{\delta S^I[\varphi]}{\delta \varphi(x)} - \frac{\delta^2 S^I[\varphi]}{\delta \varphi(y) \delta \varphi(x)} \right] + \text{const} \]

The partition function is independent from the UV cutoff
1PI effective average action RG flow

\[ e^{-W_k[J]} = Z_k[J] = e^{-\Delta S_k[\frac{\delta}{\delta J}]} Z_k[J] = \int[d\varphi] e^{-S[\varphi]-\Delta S_k[\varphi]+J\varphi} \]

Infrared regulator: \[ \Delta S_k[\varphi] = \frac{1}{2} \varphi \cdot R_k \cdot \varphi \]

Legendre transform

\[ \Gamma_k[\phi] = \text{extr} (J \cdot \phi - W_k[J]) - \Delta S_k[\phi] \]

\[ e^{-\Gamma_k[\phi]} = \int[d\varphi] e^{-S[\varphi]+\frac{\delta \Gamma_k}{\delta \varphi} \cdot (\varphi-\phi)-\Delta S_k[\varphi-\phi]} \]

Wetterich/Morris equation

\[ \partial_t \Gamma_k = \frac{1}{2} \text{Tr} \left[ \left( \Gamma_k^{(2)} + R_k \right)^{-1} \partial_t R_k \right] \quad t = \ln \frac{k}{k_0} \]

Legendre type relation between Wilsonian action and effective average action

\[ \Gamma_\Lambda[\varphi^c] + \frac{1}{2} (\varphi^c - \Phi) \cdot \Delta_H \cdot (\varphi^c - \Phi) = S_\Lambda^I[\Phi] \]
Multicritical Yukawa theories

Consider a QFT of one real scalar field and Dirac fermions \((N_f, d_\gamma)\) Symmetries: \(U(N_f)\), \(Z_2\)

Physics also for interacting fermion systems, for SUSY models/emergent susy, quark-mesonic interactions,…

We study the model at criticality with the flow of the effective average action in a local potential approximation (LPA) + (eventually) anomalous dimensions

Truncation: \(\Gamma_k [\phi, \psi, \bar{\psi}] = \int d^d x \left( \frac{1}{2} Z_{\phi, k} \partial^\mu \phi \partial_\mu \phi + V_k (\phi) + Z_{\psi, k} \bar{\psi} \gamma^\mu i \partial_\mu \psi + i H_k (\phi) \bar{\psi} \psi \right)\)

Two functions

Rescaled dimensionless quantities: \(v_k (\phi) = k^{-d} V_k \left( \frac{Z^{1/2} \phi_k}{k^{(d-2)/2}} \right), \quad h_k (\phi) = k^{-1} Z^{1/2} H_k \left( \frac{Z^{1/2} \phi_k}{k^{(d-2)/2}} \right)\)

Flow equation for linear optimised regulators

\[
\dot{v} = -d v + \frac{d - 2 + \eta_\phi}{2} \phi v' + C_d \left( \frac{1 - \eta_\phi}{1 + v''} - X_f \frac{1 - \eta_\phi}{1 + h^2} \right)
\]

\[
\dot{h} = h (\eta_\psi - 1) + \frac{d - 2 + \eta_\phi}{2} \phi h' + C_d \left[ 2h \left( h' \right)^2 \left( \frac{1 - \eta_\phi}{1 + h^2} \right) \left( 1 + v'' \right) + \frac{1 - \eta_\phi}{1 + h^2} \left( 1 + v'' \right)^2 \right] - h'' \left( 1 - \eta_\phi \right) \left( 1 + v'' \right)
\]

Symmetries: \(v\) even and \(h\) odd.

Parameters of the problem: \((d, X_f)\)
Numerical analysis

Multicritical structure dictated by the marginal interactions, analysis with canonical dimensions

\[
\phi^{2n} : \quad d_c^\phi(n \geq 2) = \frac{2n}{n-1} = 4, \frac{8}{3}, \frac{5}{2}, \frac{12}{5}, \ldots
\]

\[
\phi^{2n+1} \overline{\psi} \psi : \quad d_c^\psi(n \geq 0) = \frac{4(n+1)}{2n+1} = 4, \frac{8}{3}, \frac{12}{5}, \ldots
\]

- Numerical evolution from the origin
- Numerical evolution from the asymptotic region
- Polynomial truncations

Strategy:

From the origin \( \lambda_f = 1 \)

\[
\frac{5}{2} < d < 4
\]

Boundary conditions:

\[
v'(0) = 0 \quad h(0) = 0
\]

\[
v''(0) = \sigma \quad h'(0) = h_1
\]

These simple non linear ODEs already know pretty much!
Numerical analysis from the asymptotic region

At large field values one can construct the asymptotic expansion of the solution as a function of free parameters and then evolve numerically towards the origin imposing the known boundary conditions

\[ v'(0) = 0 \quad h(0) = 0 \]

Some properties of the fully non trivial LPA scaling solutions in d=3:
if \( X_f < 1.64 \) the scalar is in the broken phase.

- \( X_f = 1 \)

- \( X_f = 2 \)

Locus of the solutions in the plane \((\sigma, h_1)\) as function of \( X_f \)
### Polynomial truncations

\[
v(\rho) = \sum_{n=0}^{N_v} \frac{\lambda_n}{n!} \rho^n
\]

\[
h(\phi) = \phi \sum_{n=0}^{N_h-1} \frac{h_n}{n!} (\rho - \kappa)^n
\]

Expansions: around the origin or non trivial vacuum (I,II) vs numerical ODE sol.

\[
\begin{array}{ccccccc}
N_v = 9 & N_h = 8 & X_f = 1 & \\
\end{array}
\]

---

### Table IX: Case

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### Table VIII: Case

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**Strong improvement comparing to results obtained with a smaller truncation with fixed** \( h(\phi) = h_1 \phi \)

Moving to \( Z_\phi(\phi) \) and \( Z_\psi(\phi) \) not so useful, **Knorr** Phys. Rev. B94 (2016) 245102

probably needed 4 derivative expansion or momentum dependent vertex expansion.
Effective average Hamiltonian action


\[ S[p, q] = \int dt \left[ p(t)\partial_t q(t) - H(p(t), q(t)) \right] \]

\[ e^{\frac{i}{\hbar}W[I, J]} = \int [dpdq] \mu[p, q] e^{\frac{i}{\hbar}\{S[p, q] + I \cdot p + J \cdot q\}} \]

\[ \Gamma^H[\bar{p}, \bar{q}] = \text{ext}_{I, J} (W[I, J] - I \cdot \bar{p} - J \cdot q) \]

\[ e^{\frac{i}{\hbar}\Gamma^H[\bar{p}, \bar{q}]} = \int [dpdq] \mu[p, q] e^{\frac{i}{\hbar}\left\{ S[p, q] - (q - \bar{q}) \frac{\delta \Gamma^H}{\delta q} - (p - \bar{p}) \frac{\delta \Gamma^H}{\delta p} \right\}} \]

Perturbative techniques are easily extended. Wilsonian renormalization group for the action written in terms of the Hamiltonian. Define an effective Hamiltonian flow. The coarse-graining is in the full phase space.

\[ e^{iW_k[I, J]} = \int [dpdq] \mu_k[p, q] e^{i\{S[p, q] + \Delta S_k[p, q] + I \cdot p + J \cdot q\}} \quad \Delta S_k[p, q] = \frac{1}{2}(p, q) \cdot R_k \cdot (p, q)^T \]

\[ e^{i\Gamma_k[\bar{p}, \bar{q}]} = \int [dpdq] \mu_k[p, q] e^{i\{S[p, q] + \Delta S_k[p - \bar{p}, q - \bar{q}] - (p - \bar{p}) \frac{\delta \Gamma_k}{\delta p} -(q - \bar{q}) \frac{\delta \Gamma_k}{\delta q} \}} \]

Example of regulators in \((q, p)\)

\[ R_k(t, t') = \begin{pmatrix} 0 & r_k(-\partial_t^2)\partial_t\delta(t - t') \\ -r_k(-\partial_t^2)\partial_t\delta(t - t') & 0 \end{pmatrix} \quad \mu_k = \left[ \text{Det} \frac{1}{2\pi} \begin{pmatrix} 0 & (1 + r_k(-\partial_t^2)) \partial_t\delta(t - t') \\ -(1 + r_k(-\partial_t^2)) \partial_t\delta(t - t') & 0 \end{pmatrix} \right]^{\frac{1}{2}} \]

\[ R_k(t, t') = \begin{pmatrix} 0 & \mathcal{R}_k(-\partial_t^2)\delta(t - t') \\ \mathcal{R}_k(-\partial_t^2)\delta(t - t') & 0 \end{pmatrix} \quad \mu_k = \left[ \text{Det} \frac{1}{2\pi} \begin{pmatrix} \mathcal{R}_k(-\partial_t^2)\delta(t - t') & \partial_t\delta(t - t') \\ -\partial_t\delta(t - t') & \mathcal{R}_k(-\partial_t^2)\delta(t - t') \end{pmatrix} \right]^{\frac{1}{2}} \]
Flow equations

**Local Hamiltonian approximation** (constant \( \bar{q}, \bar{p} \))

**Off-diagonal** IR regulator

\[
\frac{dH_r}{dr} = -\frac{1}{2(1+r)^2} \left( \text{det} H_r^{(2)} \right)^{\frac{1}{2}}
\]

One can study the spectrum of the quantum mechanical models non quadratic in the momenta, which have a non reducible path integral.

**Diagonal** IR regulator

\[
\partial_\mathcal{R} \dot{H}_\mathcal{R} = -\frac{1}{\pi} \arctan \left( \frac{\Lambda}{\mathcal{R}} \right) + \frac{2\mathcal{R} + \partial_{\bar{p}\bar{p}} H_\mathcal{R} + \partial_{\bar{q}\bar{q}} H_\mathcal{R}}{2\pi \mathcal{D}_\mathcal{R}} \arctan \left( \frac{\Lambda}{\mathcal{D}_\mathcal{R}} \right)
\]

\[
\mathcal{D}_\mathcal{R} = \sqrt{\mathcal{R}^2 + \mathcal{R} \left( \partial_{\bar{p}\bar{p}} H_\mathcal{R} + \partial_{\bar{q}\bar{q}} H_\mathcal{R} \right) + \text{det} H_\mathcal{R}^{(2)}}
\]
A quantum mechanical example

To integrate the flow from the UV to the IR we need to specify the bare Hamiltonian at the UV scale. This is in 1-1 correspondence with Hamiltonian operator, being its Weyl symbol (i.e. Weyl ordered).

$$\langle x|\hat{O}|y \rangle = \int dp \langle x|p \rangle OW \left( p, \frac{x+y}{2} \right) \langle p|y \rangle \quad OW(p,q) = \int dx e^{ipx} \langle q - \frac{x}{2}|\hat{O}(\hat{p},\hat{q})|q + \frac{x}{2} \rangle$$

First example for $H_n(p,q) = \left( \frac{p^2 + \omega^2 q^2}{2} \right)^n$

Weyl symbol

$H_{2W}(p,q) = \left( \frac{p^2 + q^2}{2} \right)^2 - \frac{1}{4}$

$H_{3W}(p,q) = \left( \frac{p^2 + q^2}{2} \right)^3 - \frac{5}{4} \left( \frac{p^2 + q^2}{2} \right)$

From numerical evolution one gets the effective Hamiltonian.

Numerical error in the spectrum <0.1%

Diagonal cutoff schemes seem to work better.

Another example:

The Hamiltonian $H_n(p,q) = p^n + a q^n$ has instead the same Weyl symbol

Similar agreement.
Regge limit of strong interactions
Pomeron-Odderon Reggeon Field Theory

The main physical motivation is the idea that QCD, in the high energy (Regge) limit and at large distances, can be described by an effective theory such as Reggeon Field Theory (RFT), with local fields and local interactions.

- Possible transition from QCD to the RFT regime:
  - BFKL physics: fundamental gluon (and quarks) organise themselves in composite fields (of reggeized gluons) giving as leading color singlet objects interacting Pomeron and Odderon,
    - BFKL Pomeron ($J > 1$), Odderon ($J \simeq 1$) and both $\alpha' \simeq 0$
  - This should be at the “UV” boundary of RFT, below which (at larger distances) they may be considered approximately local with $J \simeq 1$ and a non zero $\alpha'$ and described by Regge poles, as in old S-matrix analysis of strong interactions intrinsically non perturbative.
  - The onset of such a transition should involve mainly perturbative physics.
  - Here we investigate some features of RFT in 2 transverse dimensions

\[ s \rightarrow \infty \]
\[ t \simeq 0 \]
QCD in the Regge limit.

In early QCD times perturbative BFKL analysis found gluon reggeization, the Pomeron, as a composite state $\psi$ of 2 reggeized gluons and later the Odderon (C,P odd), as a composite state $\chi$ of 3 reggeized gluons, solution of the BKP equation in the lowest non trivial approximation. Simple exchanges of such objects are corrected by interactions in presence of more reggeized gluons in the t channel which are necessary to unitarize the theory.

Diagrams with reggeized gluons containing PPP and POO vertices: interactions are local in rapidity but non local in transverse space.

Diagrams with reggeized gluons

$$\frac{\partial N}{\partial \tau} = KN - V_{PPP}NN + V_{POO}OO$$

$$\frac{\partial O}{\partial \tau} = KO - V_{OPO}(NO + ON)$$

Approx. evolution in rapidity

Similar objects are found in other approaches to the Regge limit of QCD: CGC, Dipole/Wilson lines.

RFT might appear at high energies (large rapidities) and large transverse distances.

**Odderon** recently in the news because of TOTEM measurements at LHC!
Strong interactions and old Regge theory

About half a century ago V.N. Gribov introduced phenomenologically the RFT. Starting point: Sommerfeld-Watson representation of the elastic scattering amplitudes.

\[ T_{AB}(s, t) = \int \frac{d\omega}{2i} \xi(\omega)s^{1+\omega}F(\omega, t). \]

\[ \xi(\omega) = \frac{\tau - e^{-i\pi\omega}}{\sin \pi\omega} \]

\[ \tau = \pm 1 \]

- **Regge pole description** in the complex \( \omega = J - 1 \) plane

- The leading pole: even signedatured **Pomeron** with vacuum quantum numbers, trajectory \( \alpha(t) \approx \alpha_0 + \alpha' t \)

- Unitarity in the crossed (t-channel): **multi pomeron states**, branch-point singularities (Regge cuts)

- Analysis of experimental inclusive cross sections in the triple Regge region showed that a **triple Pomeron interaction** should be introduced.

- In the ’70 it was conjectured that another pole with odd quantum numbers (P,C, \( \tau \)) could exist, the so called **Odderon** with \( \alpha(0) \) close to 1.

- **The Pomeron RFT was found to be in the same universality class as directed percolation.** Cardy (1980), Canet, et al. (2004), Bartels, Contreras, G.P.V. (2016),

Non perturbative FRG analysis give good results!
Interactions are constrained by signature: conservation

\[ \text{Symmetries} \quad \begin{align*} &\text{Reggeons have different signature factors,} \\
&\text{multi reggeon cut has discontinuity with overall sign from} \quad -i\Pi_j(i\xi_j) \end{align*} \]

\[ \text{Pomeron:} \xi \simeq i \quad \text{(imaginary)} \quad , \quad \text{Odderon:} \xi \simeq -\frac{2}{\pi \omega} \quad \text{(real)} \]

Couplings can be real or imaginary!

- \( n \) Pomeron t-channel states induced by interactions gets a factor \((-1)^{n-1}\) Therefore the pomeron self energy is negative.
  The triple Pomeron coupling by convention is chosen imaginary.
  Quartic Pomeron couplings are real.

- Odderon has negative signature:
  transition \( P \rightarrow OO \) is real valued; transition \( O \rightarrow OP \) is imaginary
  Quartic interactions: most coupling remain real, but
  \( O \rightarrow OOO \quad \text{and} \quad P \rightarrow P + OO \) have imaginary coupling
Local effective action for RFT

\[ \Gamma[\psi^\dagger, \psi, \chi^\dagger, \chi] = \int d^Dx \, d\tau \left( Z_P \left( \frac{1}{2} \psi^\dagger \partial_{\tau} \psi - \alpha_P \psi^\dagger \nabla^2 \psi \right) + Z_O \left( \frac{1}{2} \chi^\dagger \partial_{\tau} \chi - \alpha_O \chi^\dagger \nabla^2 \chi \right) + V_k[\psi, \psi^\dagger, \chi, \chi^\dagger] \right) \]

- Allowed cubic interactions

\[ V_3 = -\mu_P \psi^\dagger \psi + i\lambda \psi^\dagger (\psi + \psi^\dagger) \psi - \mu_O \chi^\dagger \chi + i\lambda \chi^\dagger (\psi + \psi^\dagger) \chi + \lambda_3 (\psi^\dagger \chi^2 + \chi^\dagger \psi^2) \]

- Allowed quartic interactions

\[ V_4 = \lambda_{41} (\psi \psi^\dagger)^2 + \lambda_{42} \psi^\dagger \psi (\psi^2 + \psi^\dagger^2) + \lambda_{43} (\chi \chi^\dagger)^2 + i\lambda_{44} \chi \chi^\dagger (\chi^2 + \chi^\dagger^2) + i\lambda_{45} \psi \psi^\dagger (\chi^2 + \chi^\dagger^2) + \lambda_{46} \psi^\dagger \chi^\dagger \chi + \lambda_{47} \chi^\dagger \psi^\dagger (\psi^2 + \psi^\dagger^2) \]

- ... 

- States with even and odd Odderon number do not mix.
- The couplings \( \lambda_3 \) and similarly \( \lambda_{44} \) and \( \lambda_{45} \) play a special role: they are responsible for the change of the Odderon number

We shall study the RG flow equation for a generic potential expanded as polynomial in the weak field approximation.

We shall consider a generic D dimensional transverse space but mainly work in D=2.
**RTF: construction of the flow equations**

General strategy used here for a polynomial truncation of the potential.

\[
\begin{align*}
\left[\Gamma^{(2)} + \mathbb{R}\right]^{-1} &= \left[\Gamma^{(2)}_{\text{free}} - V_{\text{int}}\right]^{-1} \\
&= G(\omega, q) + G(\omega, q)V_{\text{int}}G(\omega, q) + G(\omega, q)V_{\text{int}}G(\omega, q)V_{\text{int}}G(\omega, q) + \ldots
\end{align*}
\]

<table>
<thead>
<tr>
<th>$G_P(\omega, q)$</th>
<th>$G_O(\omega, q)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_P(\omega, q) = \begin{pmatrix} 0 &amp; (Z_P(i\omega + \alpha'Pq^2) + R_P - \mu_P)^{-1} \ 0 &amp; (Z_P(-i\omega + \alpha'Pq^2) + R_P - \mu_P)^{-1} \end{pmatrix}$</td>
<td>$G_O(\omega, q) = \begin{pmatrix} 0 &amp; (Z_O(i\omega + \alpha'Oq^2) + R_O - \mu_O)^{-1} \ 0 &amp; (Z_O(-i\omega + \alpha'Oq^2) + R_O - \mu_O)^{-1} \end{pmatrix}$</td>
</tr>
</tbody>
</table>

IR regulator for the coarse-graining:

\[\begin{align*}
R_P(q^2) &= Z_P\alpha'_P(k^2 - q^2)\Theta(k^2 - q^2), \\
R_O(q^2) &= Z_O\alpha'_O(k^2 - q^2)\Theta(k^2 - q^2) = rZ_O\alpha'_P(k^2 - q^2)\Theta(k^2 - q^2) \\
r &= \frac{\alpha'_O}{\alpha'_P}
\end{align*}\]

Anomalous dimensions:

\[\begin{align*}
\eta_P &= -\frac{1}{Z_P}\partial_t Z_P, & \eta_O &= -\frac{1}{Z_O}\partial_t Z_O \\
\zeta_P &= -\frac{1}{\alpha'_P}\partial_t \alpha'_P, & \zeta_O &= -\frac{1}{\alpha'_O}\partial_t \alpha'_O
\end{align*}\]

Dimensionless quantities:

\[\begin{align*}
\tilde{\psi} &= Z_P^{1/2}k^{-D/2}\psi, & \tilde{\chi} &= Z_O^{1/2}k^{-D/2}\chi & \tilde{V} &= \frac{V}{\alpha'_Pk^{D+2}}
\end{align*}\]

For example:

\[\begin{align*}
\tilde{\lambda} &= \frac{\lambda}{Z_P^{3/2}\alpha'_Pk^2}, & \tilde{\lambda}_{2,3} &= \frac{\lambda_{2,3}}{Z_OZ_P^{1/2}\alpha'_Pk^2}
\end{align*}\]

Classical scaling:

\[\begin{align*}
&\left(-(D+2) + \zeta_P\right)\tilde{V} + \left(\frac{D}{2} + \frac{\eta_P}{2}\right)\left(-\frac{\partial \tilde{V}}{\partial \tilde{\psi}} + \tilde{\psi}^\dagger \frac{\partial \tilde{V}}{\partial \tilde{\psi}^\dagger}\right) + \left(\frac{D}{2} + \frac{\eta_O}{2}\right)\left(\tilde{\chi}^\dagger \frac{\partial \tilde{V}}{\partial \tilde{\chi}} + \tilde{\chi} \frac{\partial \tilde{V}}{\partial \tilde{\chi}^\dagger}\right)
\end{align*}\]
Cubic truncation: beta functions

Performing the traces, the beta functions for dimensionless quantities are:

\[
\begin{align*}
\hat{\mu}_P &= (-2 + \eta_P + \zeta_P) \mu_P + 2A_P \frac{\lambda^2}{(1 - \mu_P)^2} - 2A_{Or} \frac{\lambda_3^2}{(r - \mu_O)^2} \\
\hat{\mu}_O &= (-2 + \eta_O + \zeta_P) \mu_O + 2(A_P + A_{Or}) \frac{\lambda_2^2}{(1 + r - \mu_P - \mu_O)^2} \\
\hat{\lambda} &= (-2 + D/2 + \zeta_P + \frac{3}{2} \eta_P) \lambda + 8A_P \frac{\lambda^3}{(1 - \mu_P)^3} - 4A_{Or} \frac{\lambda_3 \lambda_3^3}{(r - \mu_O)^3} \\
\hat{\lambda}_2 &= (-2 + D/2 + \zeta_P + \frac{1}{2} \eta_P + \eta_O) \lambda_2 \\
&\quad + 2\lambda_2^2 (6A_P + 5A_{Or}) + 4\lambda_3^2 (A_P + A_{Or}) - 4\lambda_2 \lambda_3^3 (A_P + 2A_{Or}) \\
&\quad + \frac{2A_P \lambda^3 (r - \mu_O)^2}{(1 + r - \mu_P - \mu_O)^3} - \frac{4A_{Or} \lambda_2 \lambda_3^3 (1 - \mu_P)^2}{(1 - \mu_P)^2 (1 + r - \mu_P - \mu_O)^3} \\
&\quad + \frac{2\lambda_2^3 (3A_P + A_{Or}) (r - \mu_O)}{(1 - \mu_P)(1 + r - \mu_P - \mu_O)^3} - \frac{4\lambda_2 \lambda_3^3 (A_P + 3A_{Or}) (1 - \mu_P)}{(r - \mu_O)(1 + r - \mu_P - \mu_O)^3} \\
\hat{\lambda}_3 &= (-2 + D/2 + \zeta_P + \frac{1}{2} \eta_P + \eta_O) \lambda_3 \\
&\quad + \frac{2\lambda_2 \lambda_3 (A_P + 2A_{Or})}{(r - \mu_O)(1 + r - \mu_P - \mu_O)^2} + \frac{4\lambda_3 \lambda_3^3 (2A_P + A_{Or})}{(1 - \mu_P)(1 + r - \mu_P - \mu_O)^2} \\
&\quad + \frac{2\lambda_2^3 \lambda_3 A_{Or}(1 - \mu_P)}{(r - \mu_O)^2 (1 + r - \mu_P - \mu_O)^2} + \frac{4\lambda_2 \lambda_3^3 A_P (r - \mu_O)}{(1 - \mu_P)^2 (1 + r - \mu_P - \mu_O)^2} \\
\hat{\nu} &= r (-\zeta_O + \zeta_P)
\end{align*}
\]

Similarly, one can find the anomalous dimensions (from the flow of 2-point functions):

\[
\begin{align*}
\eta_P &= -\frac{2A_P \lambda^2}{(1 - \mu_P)^2} + \frac{2A_{Or} \lambda_3^2}{(r - \mu_O)^3}, \\
\eta_O &= -\frac{4(A_P + A_{Or}) \lambda_3^2}{(1 + r - \mu_P - \mu_O)^3}, \\
\eta_P + \zeta_P &= -\frac{N_D \lambda^2}{D(1 - \mu_P)^3} + \frac{N_D \lambda_3^2}{D(r - \mu_O)^3}, \\
\eta_O + \zeta_O &= -\frac{4N_D \lambda_3^2}{D(1 + r - \mu_P - \mu_O)^3}.
\end{align*}
\]

\[
\]

\[
N_D = \frac{2}{\sqrt{4\pi^D \Gamma(D/2)}},
\]

\[
A_D(\eta_k, \zeta_k) = \frac{1}{D} - \frac{\eta_k + \zeta_k}{D(D + 2)}.
\]
Perturbation theory: $\epsilon$-expansion: $D = 4 - \epsilon$

Critical theory (fixed point): perturbative one loop results:

\[ \mu_P = \frac{\epsilon}{12}, \quad \lambda^2 = \frac{8\pi^2}{3}\epsilon, \quad \eta_P = -\frac{\epsilon}{6}, \quad \zeta_P = \zeta_O = \frac{\epsilon}{12}. \]

\[ \mu_O = \frac{95+17\sqrt{33}}{2304}\epsilon, \quad \lambda^2 = \frac{23\sqrt{6}+11\sqrt{22}}{48}\epsilon, \quad \lambda_3 = 0, \quad \eta_O = -\frac{7+\sqrt{33}}{72}\epsilon, \quad r = \frac{3}{16}(\sqrt{33}-1) \]

Critical exponents: two relevant directions

\[ \alpha_1 = -2 + \frac{\epsilon}{4} \rightarrow \nu_P = \frac{1}{2} + \frac{\epsilon}{16} \]

\[ \alpha_2 = -2 + \frac{\epsilon}{12} \rightarrow \nu_O = \frac{1}{2} + \frac{\epsilon}{48}. \]

The coupling of the changing Odderon number operator is zero!

The P $\rightarrow$ OO transition present in perturbative QCD is irrelevant and disappears. Suppression of high mass diffractive scattering processes.

The Pomeron sector is not affected by the presence of the Odderon!

These qualitative features are maintained at non perturbative level!
Non perturbative analysis in $D=2$

Explicit analysis at order 3,4,5 of the fixed points seems to show that the interactions changing the Odderon number are absent in the critical theory.

We perform the analysis of the fixed point up to order 9, neglecting (apart in $r$) the anomalous dimensions.

Couplings: fixed point values are stable at order 9.

We find three relevant directions.

Critical exponents: $\nu = -1/\lambda$

$\nu_P \simeq 0.73$, $\nu_O \simeq 0.6$

$\lambda_3 \simeq -0.26$

Anomalous dimensions (cubic truncation estimate, close to $\epsilon$-expansion result):

$\eta_P \simeq -0.33$, $\eta_O \simeq -0.35$ and $\zeta_P = \zeta_O \simeq +0.17$
Conclusions and outlook

- **Functional renormalization group** is a powerful tool not yet fully exploited to study both critical and off-critical QFTs.

- It can be used both at perturbative and non perturbative (wilsonian) level

- In perturbation theory it is possible to directly compare or complement results with ones from CFT techniques (conformal universal data).

- At non perturbative level one has scheme dependent exact RG flow equations.

- Main problem: choice of truncations and approximations! Still new ideas are needed for a systematic control of the convergence.

- Gauge theories still harder to investigate at accurate level

- In many cases gives results at the level of montecarlo analysis for strongly interacting theories.

- At theoretical level tool to study the (geometry of) theory space of QFTs